A strange five vertex model and multispecies ASEP on a ring

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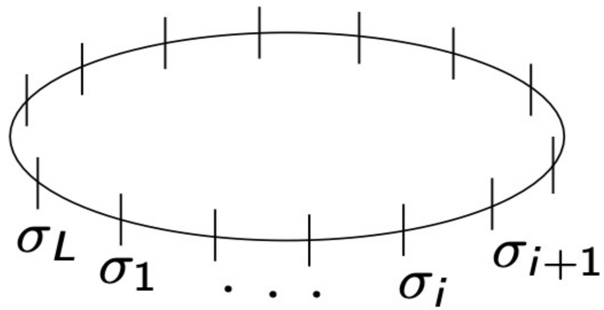
This talk focuses on algebraic and combinatorial aspects of stationary states of Asymmetric Simple Exclusion Process (ASEP) on 1D periodic lattice which become intriguing in **multispecies** setting*.*

I: n-Species Asymmetric Simple Exclusion Process (n-ASEP) and stationary states



1D periodic chain with L sites

$\sigma_i \in \{0, 1, \ldots, n\}$ (n-ASEP)

Stochastic dynamics

$$(\sigma_i, \sigma_{i+1}) \rightarrow (\sigma'_i, \sigma'_{i+1})$$

$$(\alpha, \beta) \rightarrow (\beta, \alpha) \text{ with rate } t^{\theta(\alpha>\beta)}$$

$$\theta(\text{true}) = 1, \quad \theta(\text{false}) = 0$$

$$V = \bigoplus_{\alpha=0}^{n} \mathbb{C}|\alpha\rangle \quad \text{space of one particle states}$$

$$V^{\otimes L} = \bigoplus_{0\le\sigma_1,\dots,\sigma_L\le n} \mathbb{C}|\sigma_1,\dots,\sigma_L\rangle \quad \text{space of states of } n\text{-ASEP}$$

Master equation (τ(time) $\neq t$(hopping asymmetry))

$$\frac{d}{d\tau}|P(\tau)\rangle = H|P(\tau)\rangle, \quad |P(\tau)\rangle = \sum_{\{\sigma_i\}} \underbrace{\mathbb{P}(\sigma_1,\dots,\sigma_L;\tau)}_{probability}|\sigma_1,\dots,\sigma_L\rangle$$

Markov matrix

$$H = \sum_{i\in\mathbb{Z}_L} H^{loc}_{i.i+1}, \quad H^{loc}_{i,i+1} = 1\otimes\cdots\otimes 1\otimes \underset{i,i+1}{H^{loc}}\otimes 1\otimes\cdots\otimes 1$$

Local Markov matrix

$$H^{loc}|\alpha,\beta\rangle = \begin{cases} t(|\beta,\alpha\rangle - |\alpha,\beta\rangle) & \alpha > \beta, \\ |\beta,\alpha\rangle - |\alpha,\beta\rangle & \alpha \le \beta. \end{cases}$$

Integrability (R matrix)

$$R(z): \quad V \otimes V \longrightarrow V \otimes V$$
$$|\alpha\rangle \otimes |\beta\rangle \longmapsto \sum_{0 \le \gamma, \delta \le n} R(z)^{\gamma,\delta}_{\alpha,\beta} \, |\gamma\rangle \otimes |\delta\rangle$$

$$R(z)^{\alpha,\alpha}_{\alpha,\alpha} = 1, \quad R(z)^{\alpha,\beta}_{\alpha,\beta} = \frac{(1-z)t^{\theta(\alpha<\beta)}}{1-tz}, \quad R(z)^{\beta,\alpha}_{\alpha,\beta} = \frac{(1-t)z^{\theta(\alpha>\beta)}}{1-tz} \quad (\alpha \neq \beta)$$

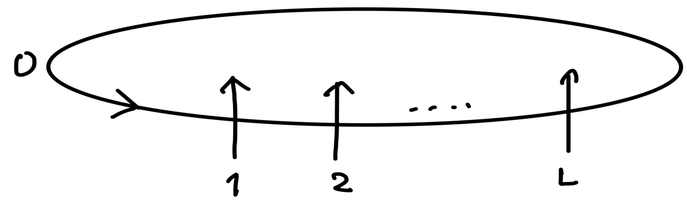
Yang-Baxter equation(YBE): $R(x)_{12}R(xy)_{13}R(y)_{23} = R(y)_{23}R(xy)_{13}R(x)_{12}$

$R(z)_{ij} = R(z)$ acting on the (i,j)-components of $V \otimes \cdots \otimes V$.

$$\left.\frac{d\check{R}(z)}{dz}\right|_{z=1} = -(1-t)H^{loc}, \text{ where } \check{R}(z) = PR(z), \quad P(u \otimes v) = v \otimes u.$$

Integrability (Transfer matrix)

$$T(z) = \mathrm{Tr}_0(R(z)_{0L} \cdots R(z)_{02} R(z)_{01}) : \; V^{\otimes L} \longrightarrow V^{\otimes L}$$



$$\mathrm{YBE} \rightarrow \quad [T(x), T(y)] = 0$$

$$T(1)^{-1} T(z) = \mathrm{Id} + \frac{1-z}{1-t} H + \mathcal{O}((z-1)^2)$$

$$T(1)|\sigma_1, \ldots, \sigma_L\rangle = |\sigma_L, \sigma_1, \ldots, \sigma_{L-1}\rangle \quad \text{cyclic shift}$$

Remark: $R(z)$ is a $U_t(\widehat{sl}_n)$ quantum R matrix in the *stochastic gauge*:

$$\sum_{\gamma,\delta} R(z)^{\gamma,\delta}_{\alpha,\beta} = 1$$

H acts on each *sector* specified by particle multiplicity $\mathbf{m} = (m_0, m_1, \ldots, m_n)$

$$W(\mathbf{m}) = \bigoplus_{\sigma_1,\ldots,\sigma_L} \mathbb{C}|\sigma_1, \ldots, \sigma_L\rangle \subset V^{\otimes L},$$

where the sum extends over the configurations satisfying $\#\{\sigma_i = k\} = m_k$ $(k = 0, \ldots, n)$.

By the definition, $m_0 + m_1 + \cdots + m_n = L$. We assume $\forall m_i \geq 1$ throughout.

Stationary states

For each $\mathbf{m}$, there is a unique state $|\overline{P}(\mathbf{m})\rangle \in W(\mathbf{m})$ up to normalization such that

$$H|\overline{P}(\mathbf{m})\rangle = 0.$$

$$|\overline{P}(\mathbf{m})\rangle = \sum_{\{\sigma_i\}} \underbrace{\mathbb{P}(\sigma_1, \ldots, \sigma_L)}_{\substack{\text{stationary} \\ \text{probability}}} |\sigma_1, \ldots, \sigma_L\rangle =: \sum_{\boldsymbol{\sigma}} \mathbb{P}(\boldsymbol{\sigma})|\boldsymbol{\sigma}\rangle$$

In what follows, we consider the unnormalized stationary probability, disregarding $\sum_{\boldsymbol{\sigma}} \mathbb{P}(\boldsymbol{\sigma}) = 1$.

$n=1$: Stationary states are uniform, i.e., $\mathbb{P}(\boldsymbol{\sigma})$ is independent of $\boldsymbol{\sigma}$.

Examples of (unnormalized) stationary states

$n=2:$

$$|\mathbb{P}(1,1,1)\rangle = (2+t)|012\rangle + (1+2t)|021\rangle + \text{cyc},$$

$$|\mathbb{P}(1,2,1)\rangle = (2+t+t^2)|0112\rangle + (1+t)^2|1012\rangle + (1+t+2t^2)|1102\rangle + \text{cyc},$$

$$\begin{aligned}|\mathbb{P}(1,2,2)\rangle = {} & (3+t+t^2)|11220\rangle + (2+2t+t^2)|12120\rangle + (1+3t+t^2)|12210\rangle \\ & + (2+t+2t^2)|21120\rangle + (1+2t+2t^2)|21210\rangle + (1+t+3t^2)|22110\rangle + \text{cyc},\end{aligned}$$

where cyc denotes the terms obtained by cyclic shifts.

$n=3:$

$$\begin{aligned}|\mathbb{P}(1,1,1,1)\rangle = {} & (9+7t+7t^2+t^3)|0123\rangle + (3+11t+5t^2+5t^3)|0213\rangle \\ & + 3(1+t)^3|1023\rangle + (5+5t+11t^2+3t^3)|1203\rangle \\ & + 3(1+t)^3|2013\rangle + (1+7t+7t^2+9t^3)|2103\rangle + \text{cyc}.\end{aligned}$$

For n $\geq$ 2, stationary states are non-trivial even at t=0 (TASEP).

Matrix product construction of stationary probability

Lemma. If the operators $X_0(z),\dots,X_n(z)$ satisfy the Zamolodchikov-Faddeev (ZF) algebra

$$X_\alpha(y)X_\beta(x)=\sum_{\gamma,\delta=0}^{n} R(y/x)^{\beta,\alpha}_{\gamma,\delta}X_\gamma(x)X_\delta(y),$$

then a matrix product formula

$$\mathbb{P}(\sigma_1,\dots,\sigma_L)=\mathrm{Tr}(X_{\sigma_1}\cdots X_{\sigma_L})\qquad (X_\alpha=X_\alpha(z=1))$$

is valid provided that the trace is nonzero and finite.

$$\because)\quad H|\overline{P}(\mathbf{m})\rangle=\sum_{i\in\mathbb{Z}_L}\sum_{\boldsymbol{\sigma}}\mathbb{P}(\dots,\sigma_i,\sigma_{i+1},\dots)H^{loc}_{i,i+1}|\dots,\sigma_i,\sigma_{i+1},\dots\rangle$$

$$=\sum_{i\in\mathbb{Z}_L}\sum_{\boldsymbol{\sigma}}\sum_{\sigma'_i,\sigma'_{i+1}}\mathrm{Tr}(\cdots X_{\sigma_i}X_{\sigma_{i+1}}\cdots)\check{R}'(1)^{\sigma'_i,\sigma'_{i+1}}_{\sigma_i,\sigma_{i+1}}|\dots,\sigma'_i,\sigma'_{i+1},\dots\rangle/(t-1)$$

$$=\sum_{\boldsymbol{\sigma}}\sum_{i\in\mathbb{Z}_L}\mathrm{Tr}\Big(\cdots\big(\sum_{\sigma'_i,\sigma'_{i+1}}R'(1)^{\sigma_{i+1},\sigma_i}_{\sigma'_i,\sigma'_{i+1}}X_{\sigma'_i}X_{\sigma'_{i+1}}\big)\cdots\Big)|\dots,\sigma_i,\sigma_{i+1},\dots\rangle/(t-1).$$

Derivative of the ZF relation and $\check{R}(1)=\mathrm{id}$ leads to

$$\text{Red part}=X'_{\sigma_i}(1)X_{\sigma_{i+1}}-X_{\sigma_i}X'_{\sigma_{i+1}}(1).\quad\square$$

Constructions of the stationary states of multispecies ASEP

Algebraic	Combinatorial
Matrix product operators X_α	Multiline queue method
Prolhac-Evans-Mallick 2009	t=0: Ferrari-Martin (FM) 2009
	t=t: Martin 2020
Representations of ZF algebra	(t,q): Corteel-Mandelshtam-Williams 2022
Cantini-de Gier-Wheeler 2015	
(Application to Macdonald poly.)	(Application to Macdonald poly.)
t=0: ZF alg. from tetrahedron eq.	**t=0: FM algorithm from quantum groups**
K-Maruyama-Okado 2016	K-Maruyama-Okado 2015

The key in the KMO approaches was a 5 vertex model whose Boltzmann weights are taken from **t-deformed oscillator algebra at t=0**.

The key in this talk is yet another 5 vertex model associated with the t-oscillator algebra for $t \neq 0$, which obeys a strange weight conservation rule.

It clarifies the relation of the matrix product & multiline queue methods and refines their derivations.

II. A strange five vertex model

b; $i \to a$; j	0; $0 \to 0$; 0	0; $1 \to 1$; 1	1; $0 \to 0$; 1	0; $0 \to 1$; 1	0; $1 \to 0$; 0
S_{ij}^{ab}	1	1	$\mathbf{k}$	$\mathbf{a}^{-}$	$\mathbf{a}^{+}$

2 state model; a,b,i,j = 0,1. **Strange weight conservation rule a+b=j.**

(cf. Usual weight conservation: $S_{ij}^{ab} = 0$ unless $a + b = i + j$.)

$\mathbf{a}^{+}, \mathbf{a}^{-}, \mathbf{k}$ are generators of t-oscillator algebra:

$$\mathbf{k}\,\mathbf{a}^{\pm} = t^{\pm 1}\mathbf{a}^{\pm}\mathbf{k}, \qquad \mathbf{a}^{-}\mathbf{a}^{+} = 1 - t\mathbf{k}, \qquad \mathbf{a}^{+}\mathbf{a}^{-} = 1 - \mathbf{k}.$$

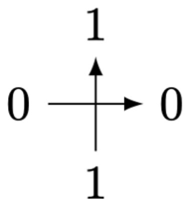
A natural representation on a bosonic Fock space:

$$F := \bigoplus_{d=0}^{\infty} \mathbb{Q}(t)|d\rangle \qquad \mathbf{k}|d\rangle = t^{d}|d\rangle, \quad \mathbf{a}^{+}|d\rangle = |d+1\rangle, \quad \mathbf{a}^{-}|d\rangle = (1-t^{d})|d-1\rangle.$$

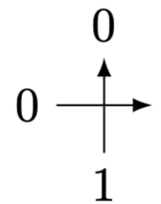
We will also use the number operator $\mathbf{h}$ defined by $\mathbf{h}|d\rangle = d|d\rangle$ so that $\mathbf{k} = t^{\mathbf{h}}$.

This ket vector is distinct from the one used to specify an ASEP local state.

Quantum picture: t-oscillator weighted 2D five vertex model



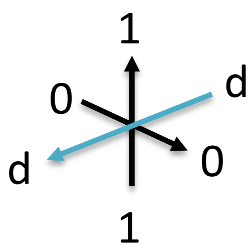
$|d\rangle = t^d|d\rangle$



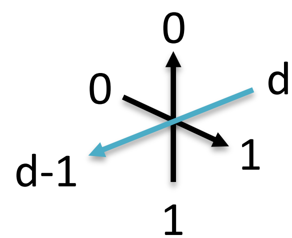
$|d\rangle = (1-t^d)|d-1\rangle$

etc

Classical picture: 3D vertex model



$= \delta_{d,d'}t^d$



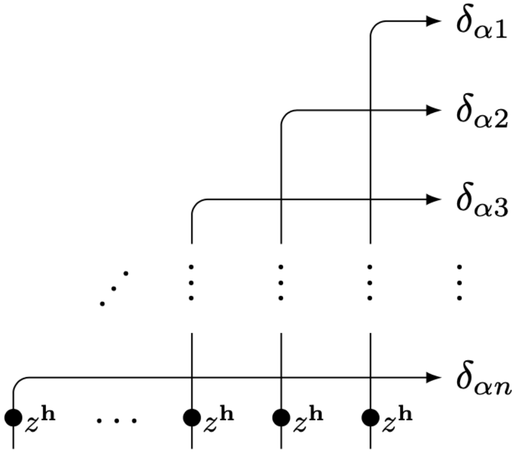
$= \delta_{d-1,d'}(1-t^d)$

etc

From now on, each 2D vertex i should be understood as carrying an arrow, perpendicular to it, with its own Fock space F running along the arrow, on which a copy of the t-oscillators $\mathbf{k}_i, \mathbf{a}_i^+, \mathbf{a}_i^-$ act.

Define

$X_\alpha(z) =$



= Partition function of the NW quadrant

Boundary condition { Right: fixed
Bottom: free

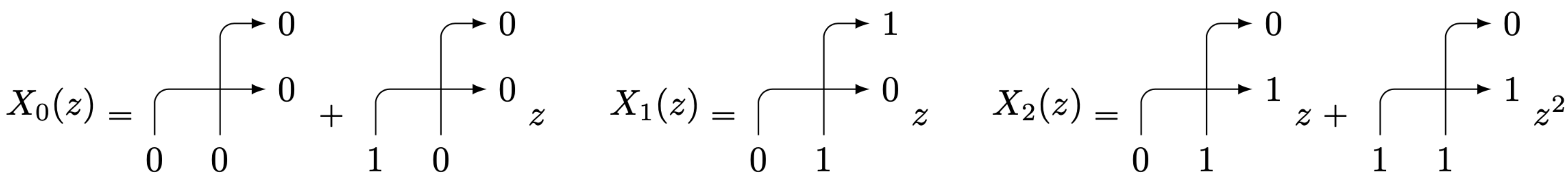
$\bullet = z^{\mathbf{h}} = 1 \text{ or } z \qquad (0 \leq \alpha \leq n)$

Can be viewed as a **Corner Transfer Matrix(CTM)** (see [Baxter, Chap.13]) of the strange five vertex model.

In the classical picture, it is a **layer transfer matrix** of size n for a 3D vertex model defined on a triangular prism.

It is a **wiring diagram** for the longest element of the symmetric group S_n.

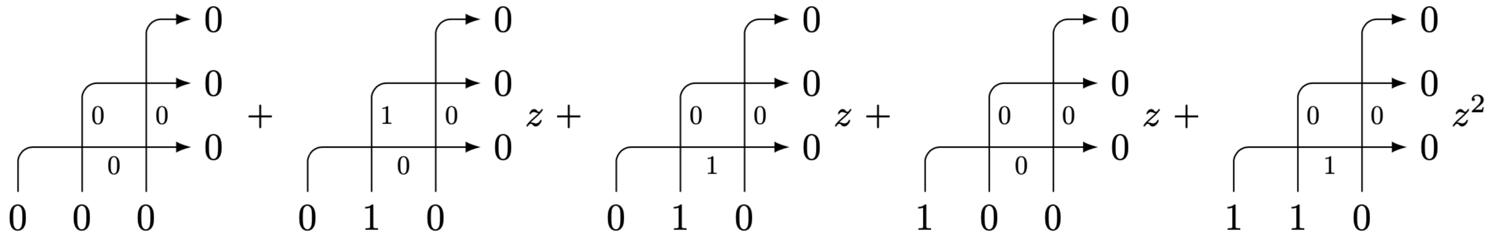
n=2 case:



$= 1 + z\mathbf{a}^+, \qquad = z\mathbf{k}, \qquad = z\mathbf{a}^- + z^2.$

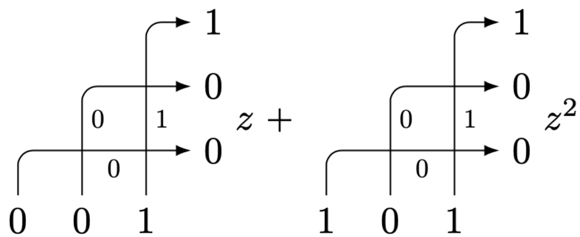
n=3 case:

$X_0(z) =$



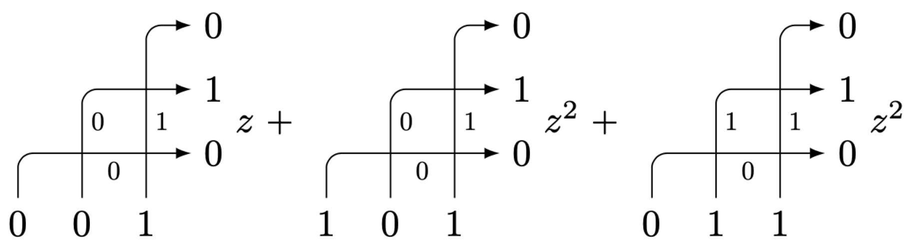
$$= 1 + z\mathbf{a}_1^+\mathbf{k}_3 + z\mathbf{a}_2^+\mathbf{a}_3^- + z\mathbf{a}_3^+ + z^2\mathbf{a}_2^+,$$

$X_1(z) =$



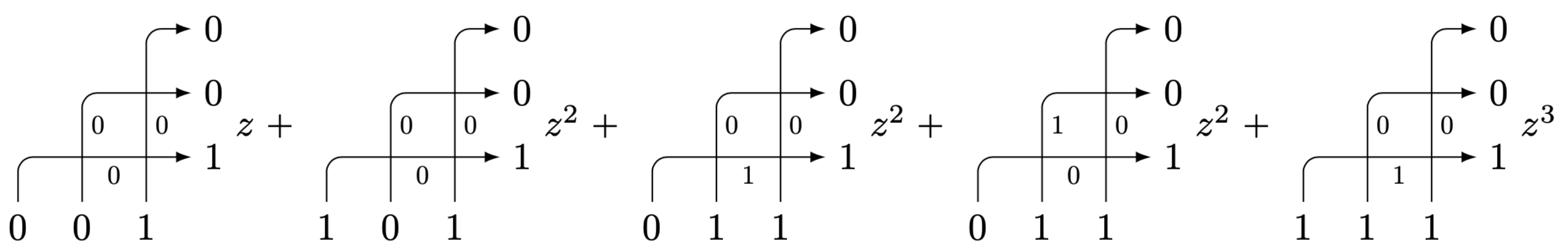
$$= z\mathbf{k}_1\mathbf{k}_2 + z^2\mathbf{k}_1\mathbf{k}_2\mathbf{a}_3^+,$$

$X_2(z) =$



$$= z\mathbf{a}_1^-\mathbf{k}_2 + z^2\mathbf{a}_1^-\mathbf{k}_2\mathbf{a}_3^+ + z^2\mathbf{k}_2\mathbf{k}_3,$$

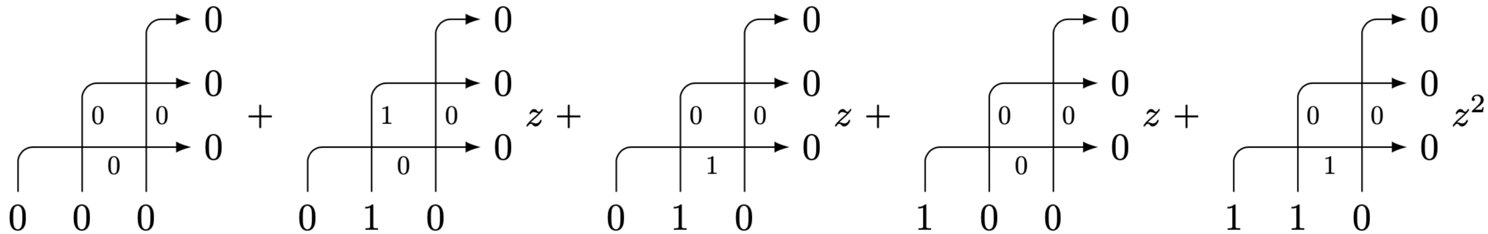
$X_3(z) =$



$$= z\mathbf{a}_2^- + z^2\mathbf{a}_2^-\mathbf{a}_3^+ + z^2\mathbf{a}_3^- + z^2\mathbf{a}_1^+\mathbf{a}_2^-\mathbf{k}_3 + z^3.$$

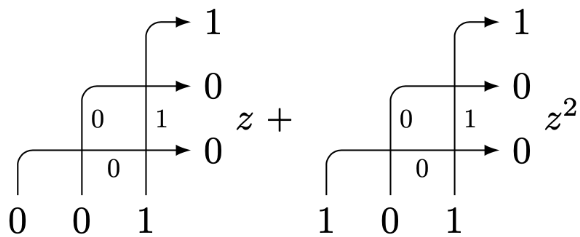
n=3 case:

$X_0(z) =$

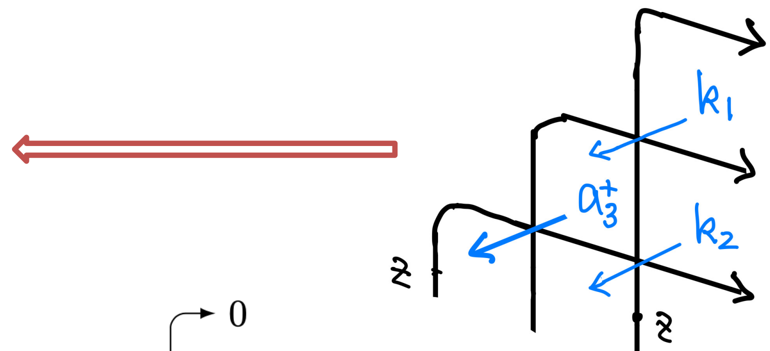


$= 1 + z\mathbf{a}_1^+\mathbf{k}_3 + z\mathbf{a}_2^+\mathbf{a}_3^- + z\mathbf{a}_3^+ + z^2\mathbf{a}_2^+,$

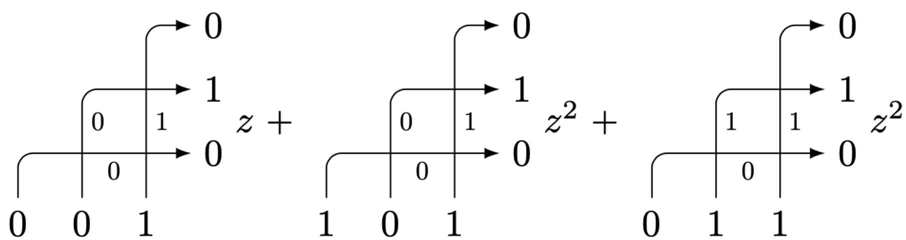
$X_1(z) =$



$= z\mathbf{k}_1\mathbf{k}_2 + z^2\mathbf{k}_1\mathbf{k}_2\mathbf{a}_3^+,$

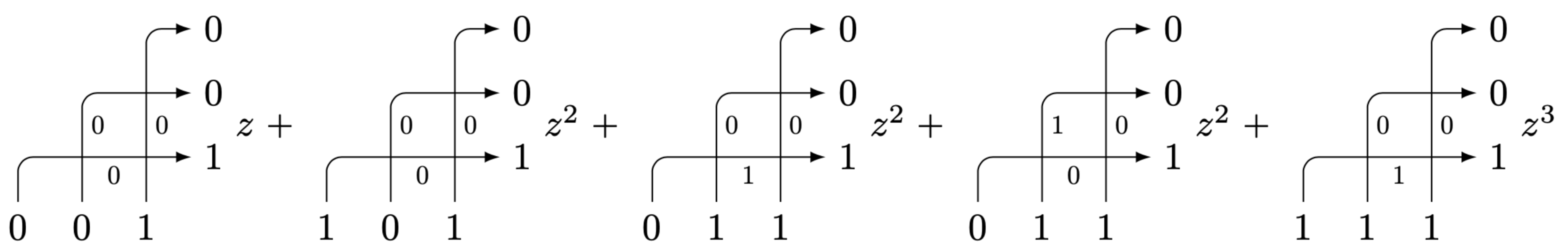


$X_2(z) =$



$= z\mathbf{a}_1^-\mathbf{k}_2 + z^2\mathbf{a}_1^-\mathbf{k}_2\mathbf{a}_3^+ + z^2\mathbf{k}_2\mathbf{k}_3,$

$X_3(z) =$

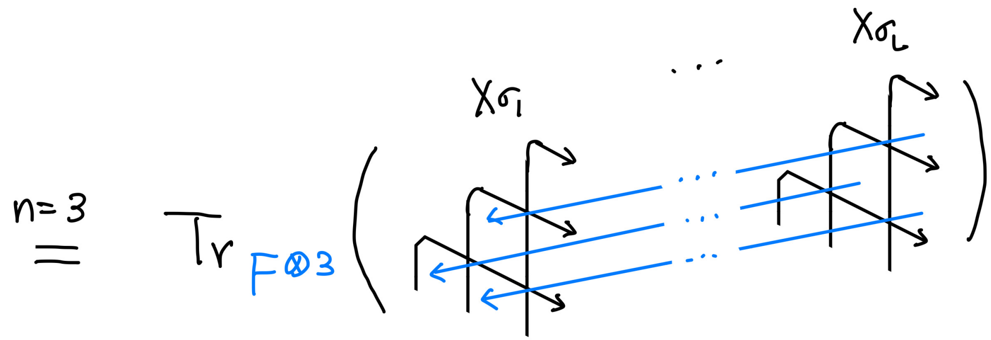


$= z\mathbf{a}_2^- + z^2\mathbf{a}_2^-\mathbf{a}_3^+ + z^2\mathbf{a}_3^- + z^2\mathbf{a}_1^+\mathbf{a}_2^-\mathbf{k}_3 + z^3.$

Theorem. $X_0(z), \ldots, X_n(z)$ satisfy the ZF-algebra relation.

Corollary. (Unnormalized) stationary probability is given by

$$\mathbb{P}(\sigma_1, \ldots, \sigma_L) = \mathrm{Tr}(X_{\sigma_1} \cdots X_{\sigma_L}) \qquad (X_\alpha = X_\alpha(z=1))$$



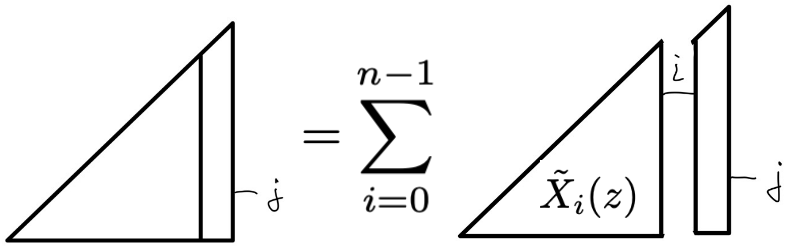
= Partition function of a 3D vertex model on a triangular prism whose boundary condition is specified according to $\sigma_1, \ldots, \sigma_L$.

Up to convention, $X\alpha(z)$ reproduces the one in Cantini–de Gier–Wheeler (2015), where the ZF-algebra was shown by combining a few lemmas.

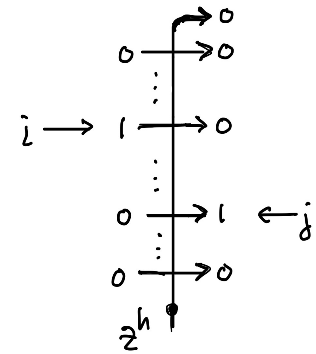
The diagram rep. for $X\alpha(z)$ based on the 5 vertex model here is the simplest one devised to date. On the next page, we present key ingredients of our proof, which makes use of the diagram and elucidate an intrinsic connection to the quantum group theory.

n-reducing recursion relation = immediate consequence of the CTM diagram

$$X_j(z) = \sum_{i=0}^{n-1} \tilde{X}_i(z)\mathcal{T}_{ij}(z)$$



$\mathcal{T}_{ij}(z) =$

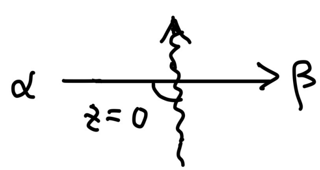


The factor $\mathcal{T}_{ij}(z)$ is linked with the quantum group theory via

$$\mathcal{L}_{\alpha,\beta} = T(z)_{\alpha,\beta+1}(\mathbf{a}_n^-)^{\delta_{\beta n}}(z^{-1}\mathbf{k}_n)^{\theta(\beta\neq n)} \quad (0 \leqslant \alpha < n,\ 0 \leqslant \beta \leqslant n),$$

where $\mathcal{L}_{\alpha,\beta}$ is a special value of a stochastic R matrix of $U_t(\widehat{sl}_n)$ on $V \otimes F^{\otimes n}$ (K-Mangazeev-Maruyama-Okado, 2016):

$\mathcal{L}_{\alpha,\beta} =$



$$= \begin{cases} \mathbf{k}_{\beta+1}\cdots\mathbf{k}_n & (\alpha = \beta) \\ \mathbf{a}_\alpha^+\mathbf{a}_\beta^-\mathbf{k}_{\beta+1}\cdots\mathbf{k}_n & (\alpha < \beta) \\ 0 & (\alpha > \beta) \end{cases} \qquad (0 \leq \alpha, \beta \leq n)$$

in "Holstein-Primakov representation".

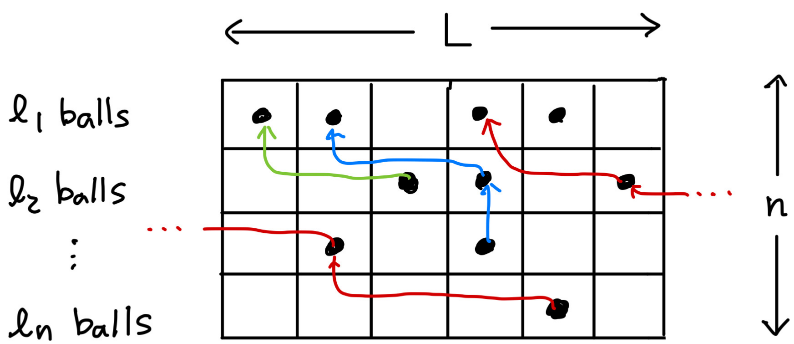
III. Relation to multiline queue construction

Multiline queue (MLQ)
Q = (ball table, pairing)

$\stackrel{\pi}{\longmapsto}$ $W(\mathbf{m})$ (ASEP states)

$\mathbf{m} = (m_0, \ldots, m_n)$

$Q =$



$\longmapsto$ $\mathrm{wt}_{q,t}(Q)|\sigma_1(Q), \ldots, \sigma_L(Q)\rangle$

rational function of q, t

$\ell_i = m_i + m_{i+1} + \cdots m_n$

MLQ construction (Martin 2020, Corteel-Mandelshtam-Williams 2022)

$$|\overline{P}(\mathbf{m})\rangle = \sum_{Q:\mathrm{MLQ}} \pi(Q)\,|_{q=1}$$

Warm up: $t = 0$ (TASEP) case

$$\mathrm{wt}_{q,t}(Q) \xrightarrow{t\to 0} \begin{cases} 1 & \text{pairing} = \text{the unique one determined by the Ferrari-Martin algorithm} \\ 0 & \text{otherwise} \end{cases}$$

Therefore, pairing degrees of freedom are suppressed for TASEP.

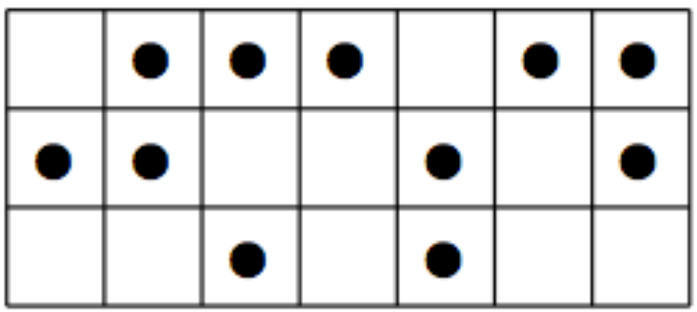
Uniform measure on ball tables
MLQ= (ball table, ~~pairing~~) $\overset{\pi}{\longmapsto}$ Stationary measure of n-TASEP

$\uparrow t \to 0$ $\qquad\qquad$ $\uparrow t \to 0$

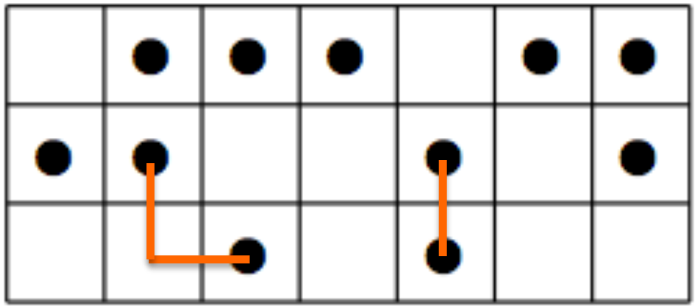
Measure on MLQ= (ball table, pairing)
biased by $\mathrm{wt}_{q,t}$ depending also on pairing $\overset{\pi}{\longmapsto}$ Stationary measure of n-ASEP

The next few pages illustrate the Ferrari-Martin algorithm (2009).
It consists of **n rounds**.

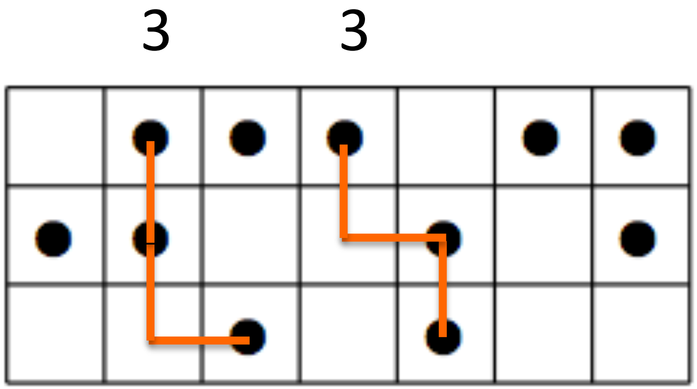
An example of ball table with n=3.



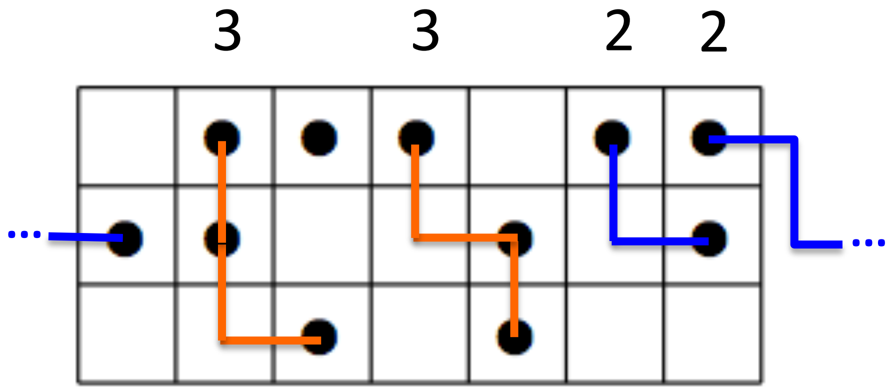
Balls in the bottom row are paired with balls in the middle row.
The partner balls are the first ones encountered either directly above or to the left, searched cyclically.



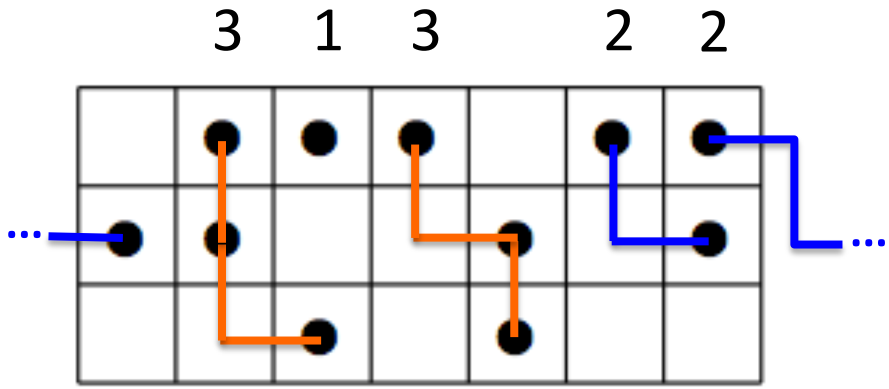
The pairings continue until they reach the top row.
The final positions, originating from the third underground level, are numbered 3.
This completes the first round of the algorithm.



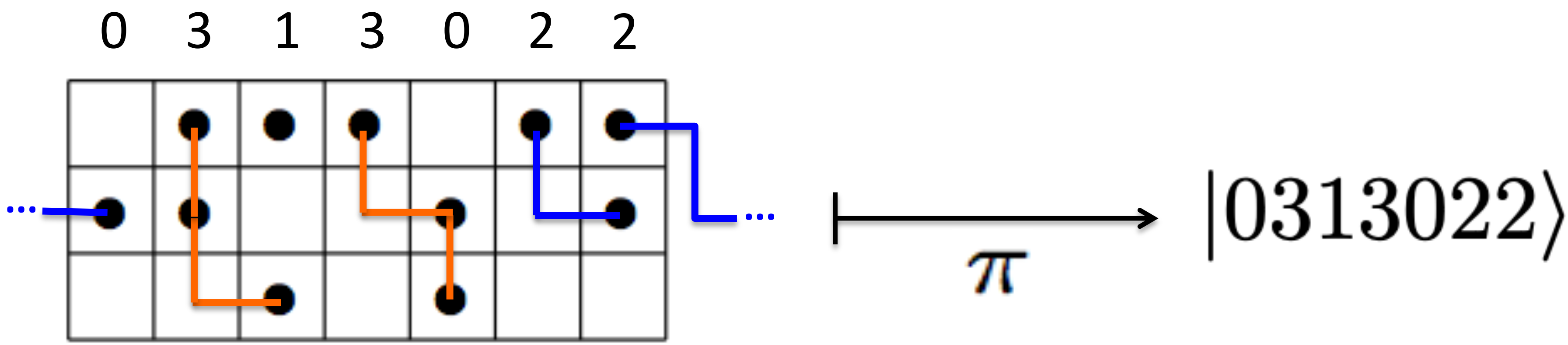
One repeats the same procedure for the remaining unpaired balls in the middle row. The final positions, originating from the second underground level, are numbered 2. This completes the second round of the algorithm.



The remaining unpaired balls in the top row are numbered 1, which is the third round of the algorithm.



The vacant slots in the top row are numbered 0.
Reading the resulting numbers forms the image of the map π.



n=3 ball table

3-TASEP state
(monomial with coefficient 1)

Security check points
Check-in counters
Customers (passengers)

The Ferrari-Martin algorithm originates in combinatorial R, a prominent example of set-theoretical solutions to YBE in Kashiwara's crystal base theory of quantum groups.(KMO15)

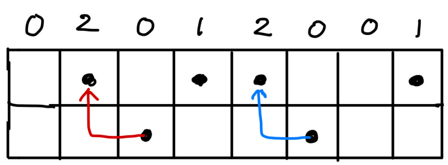
$t \neq 0$ ASEP case

Multiline queue (MLQ) $\stackrel{\pi}{\longmapsto}$ $W(\mathbf{m})$ (ASEP states)
= (ball table, pairing)

Pairing for a given ball table is no longer unique, and the ASEP state obtained as the image of π acquires a coefficient, called the weight.

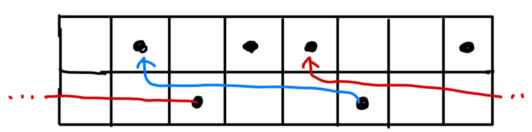
$n = 2$ example $\qquad\qquad \mathbf{m} = (4, 2, 2)$

$Q_1 =$



$$\longmapsto \underbrace{\frac{(1-t)qt^2}{1-qt^4}\,\frac{(1-t)t}{1-qt^3}}_{\mathrm{wt}_{q,t}(Q_1)} |02012001\rangle$$

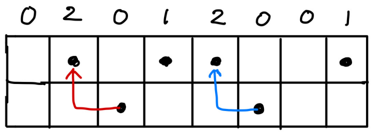
$Q_2 =$



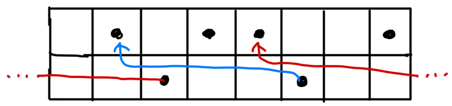
$$\longmapsto \underbrace{\frac{1-t}{1-qt^4}\,\frac{1-t}{1-qt^3}}_{\mathrm{wt}_{q,t}(Q_2)} |02012001\rangle$$

where $\mathrm{wt}_{q,t}(Q)$ denotes a *weight* of a MLQ defined combinatorially.

$n = 2$ example $\qquad\qquad \mathbf{m} = (4, 2, 2)$

$Q_1 =$ 

$$\longmapsto \underbrace{\frac{(1-t)qt^2}{1-qt^4}\,\frac{(1-t)t}{1-qt^3}}_{\mathrm{wt}_{q,t}(Q_1)}\,|02012001\rangle$$

$Q_2 =$ 

$$\longmapsto \underbrace{\frac{1-t}{1-qt^4}\,\frac{1-t}{1-qt^3}}_{\mathrm{wt}_{q,t}(Q_2)}\,|02012001\rangle$$

Define $\quad M(q,t)^{\mathbf{a},\mathbf{b}}_{\mathbf{i},\mathbf{j}} := \sum_{Q:\mathrm{MLQ}} \mathrm{wt}_{q,t}(Q)$ **vanishing unless a + b = j**

= Generating sum of MLQ weights, where dependence on **a, b, i, j** is specified by

Queueing process interpretation

Service

j = balls upstairs =(01011001)

Unused service

b = unconnected balls upstairs =(00010001)

Arriving customers

i = balls downstairs =(00100100)

Used service (Departing customers)

a = connected balls upstairs =(01001000)

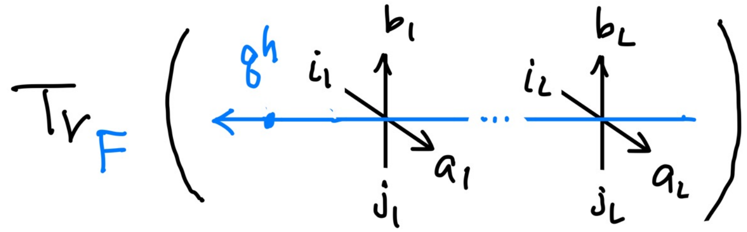
Notation

$$\mathbf{b} = (b_1, \ldots, b_L) \in \{0,1\}^L$$

$$|\mathbf{b}| = b_1 + \cdots + b_L, \quad \text{etc.}$$

Define $$S(q,t)^{\mathbf{a},\mathbf{b}}_{\mathbf{i},\mathbf{j}} := (1-qt^{|\mathbf{j}|-|\mathbf{i}|})\,\mathrm{Tr}\big(q^{\mathbf{h}} S^{a_1 b_1}_{i_1 j_1} \cdots S^{a_L b_L}_{i_L j_L}\big)$$

= BBQ stick with X shape sausages



This is also vanishing unless a + b = j.

Theorem. $$M(q,t)^{\mathbf{a},\mathbf{b}}_{\mathbf{i},\mathbf{j}} = S(q,t)^{\mathbf{a},\mathbf{b}}_{\mathbf{i},\mathbf{j}}$$

Example case: $$M(q,t)^{\mathbf{a},\mathbf{b}}_{\mathbf{i},\mathbf{j}} = \frac{(1-t)qt^2}{1-qt^4}\frac{(1-t)t}{1-qt^3} + \frac{1-t}{1-qt^4}\frac{1-t}{1-qt^3} = \frac{(1-t)^2(1+qt^3)}{(1-qt^4)(1-qt^3)},$$

$$\begin{aligned} S(q,t)^{\mathbf{a},\mathbf{b}}_{\mathbf{i},\mathbf{j}} &= (1-qt^2)\,\mathrm{Tr}(q^{\mathbf{h}} S^{00}_{00} S^{10}_{01} S^{00}_{10} S^{01}_{01} S^{10}_{01} S^{00}_{10} S^{00}_{00} S^{01}_{01}) \\ &= (1-qt^2)\,\mathrm{Tr}(q^{\mathbf{h}} \mathbf{a}^- \mathbf{a}^+ \mathbf{k}\, \mathbf{a}^- \mathbf{a}^+ \mathbf{k}) \\ &= (1-qt^2)\sum_{d\geq 0} q^d(1-t^{d+1})t^d(1-t^{d+1})t^d = \frac{(1-t)^2(1+qt^3)}{(1-qt^4)(1-qt^3)}. \end{aligned}$$

A messy sum over the pairings is consolidated into a single BBQ stick (=Trace).
What is 'created' or 'annihilated' by t-oscillator algebra are the customers in the queue.

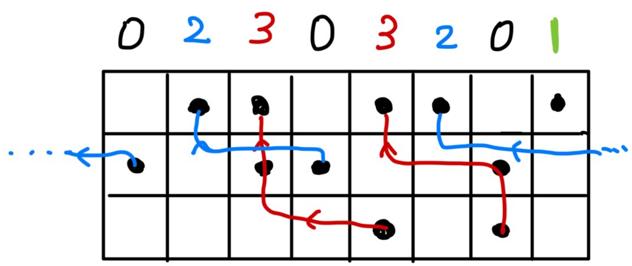
What about n ≥ 3?

n=3 example

MLQ

3-ASEP state

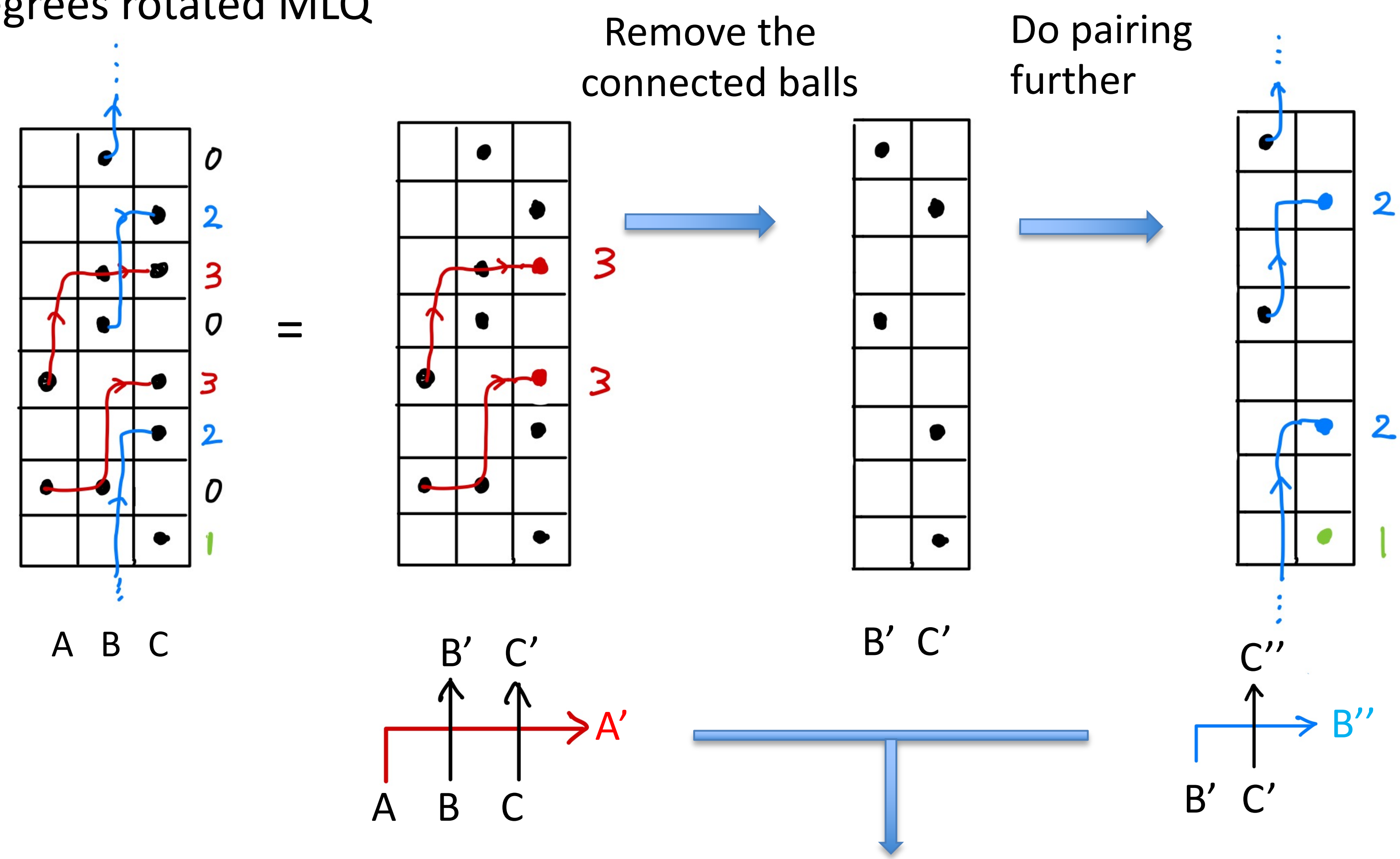
$Q =$



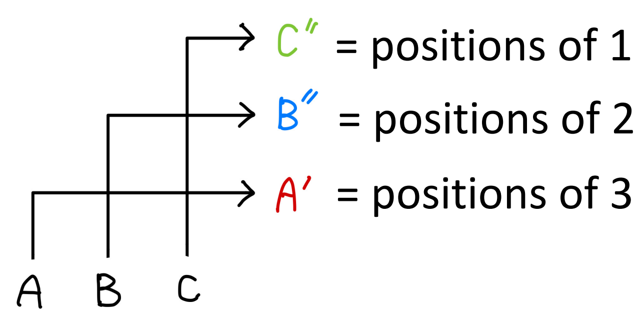
$$\longmapsto \mathrm{wt}_{q,t}(Q)|02303201\rangle$$

The MLQ construction for n ≥ 3 is a composition of the n=2 case in a "CTM manner" as illustrated in the next page.

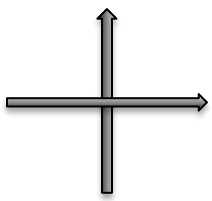
90 degrees rotated MLQ



The two diagrams combine into a *single* diagram, which becomes a CTM.

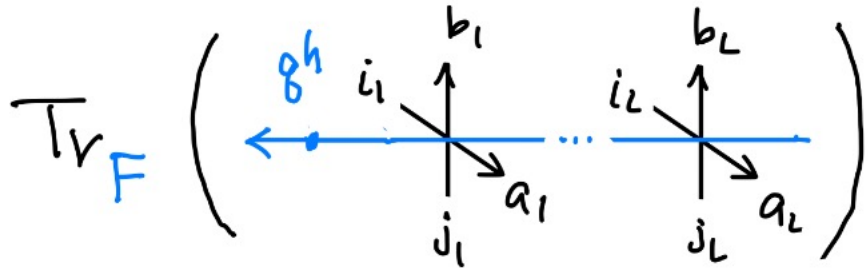


The vertices in these diagrams represent $M(q,t)^{\mathbf{a},\mathbf{b}}_{\mathbf{i},\mathbf{j}}$.
The above theorem identifies it with $S(q,t)^{\mathbf{a},\mathbf{b}}_{\mathbf{i},\mathbf{j}}$.
Making the substitution



Vertex encoding MLQ weights

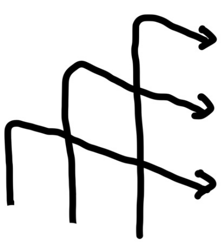
Th
=



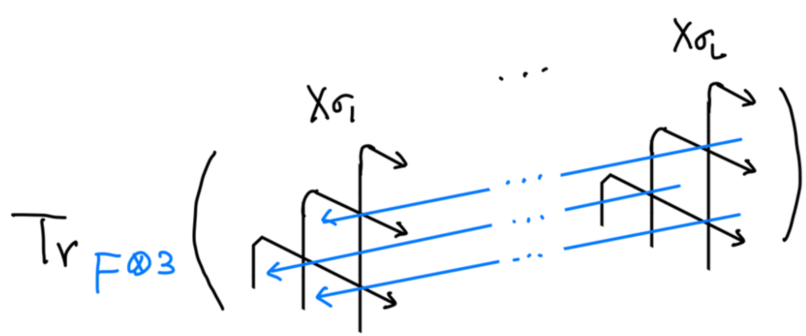
BBQ stick made of the strange 5 vertex model

and setting $q = 1$, one reproduces the matrix product formula for stationary probabilities, where each layer is a CTM of the strange 5 vertex model ($n = 3$ example shown).

$\mathbb{P}(\sigma_1, \ldots, \sigma_L)$
from MLQ



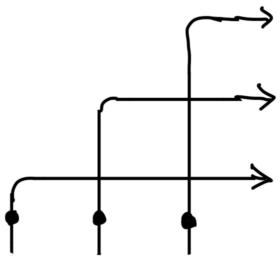
=



$$= \mathrm{Tr}(X_{\sigma_1} \cdots X_{\sigma_L})$$

IV: Summary

$X_\alpha(z) = \sum$



Corner
Transfer
Matrix

Strange 5V model

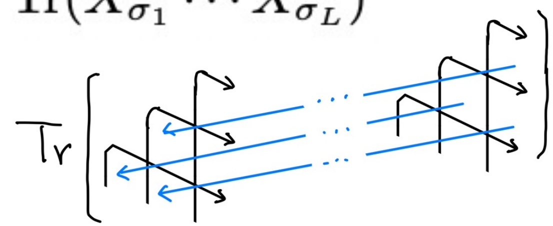
Holstein-Primakov rep. of
a $U_t(\widehat{sl}_n)$ quantum R mat.

ZF-algebra

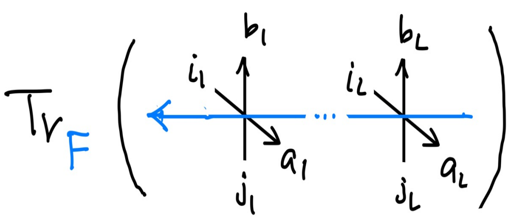
$$X(y)X(x) = \sum R\left(\frac{y}{x}\right)X(x)X(y)$$

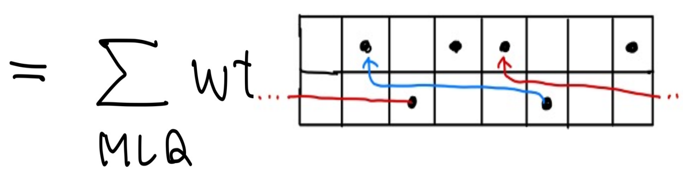
Matrix product formula with 3D interpretation

$$\mathbb{P}(\boldsymbol{\sigma}) = \mathrm{Tr}(X_{\sigma_1} \cdots X_{\sigma_L})$$



BBQ stick





Generating sum of MLQ weights

MLQ construction